0 Calculus Overview

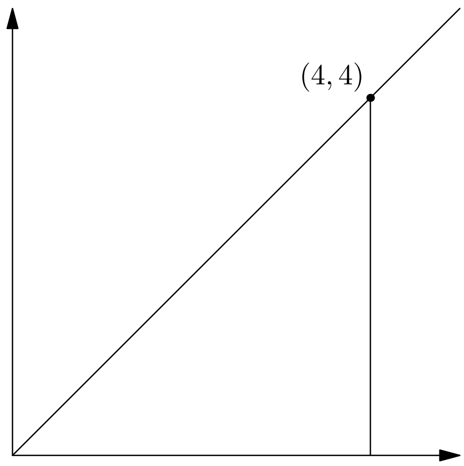
In this handout, we'll quickly outline the basics of calculus that will be necessary for this course. For those already familiar with calculus, feel free to skim through or skip the handout.

0.1 Limits and Continuity

Limits are really important tools in physics as they allow us to find the behavior of various physical systems at their endpoints (or their "limits") which are often more intuitive than the general behavior of the system. We won't really be going into the technical mathematical definition of what a limit is here, but we will instead focus on attaining a good intuitive understanding of what a limit is. A limit is basically a value that a function approaches at a certain point. For example, if $f(x) = x$, then

$$\lim_{x \to 4} f(x) = 4.$$

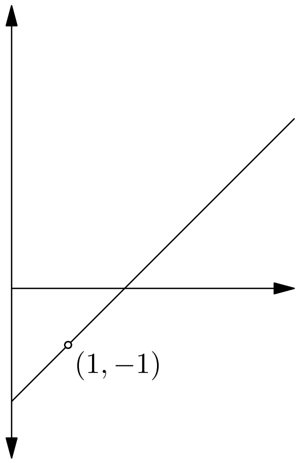
We can graphically represent this as well:



As we just saw, for a value a in the domain of f, $\lim_{x \to a} f(x) = f(a)$. However, limits can be defined for values not in the domain of f. For example, consider the function

$$f(x) = \frac{x^2 - 3x + 2}{x - 1}.$$

Note that $x^2 - 3x + 2 = (x - 1)(x - 2)$, so the graph of $f(x)$ is the same as $x - 2$, except with a "hole" located at $x = 1$, where $f(x)$ is not defined.



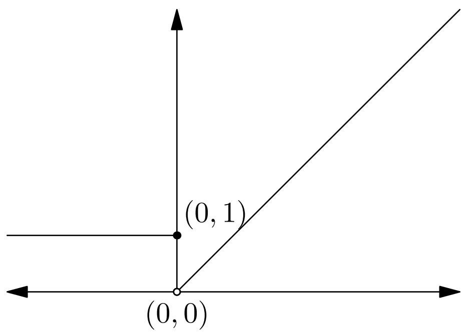
Hence, even though $f(1)$ is undefined, we have that

$$\lim_{x \to 1} f(x) = -1,$$

since f is defined near and still approaches $(1, -1)$, even though it is not defined at that point specifically.

Note that $\lim_{x \to a}$ implies that the limit is two-sided, and equal on both sides. For some functions, this might not be the case. consider the function

$$f(x) = \begin{cases} x, & x > 0 \\ 1, & x \leq 0. \end{cases}$$



We denote $\lim_{x \to a^+}$ as the limit from the positive side, and likewise $\lim_{x \to a^-}$ as the limit from the negative side. Thus, we have

$$\lim_{x \to 0^+} f(x) = 0,$$
$$\lim_{x \to 0^-} f(x) = 1,$$
$$f(0) = 1$$

from the graph above. Since $\lim_{x \to 0^+} f(x) \neq \lim_{x \to 0^-} f(x)$, we have that $\lim_{x \to 0} f(x)$ is undefined. From this, we can see that limits give us a precise way to define continuity.

Definition 0.1. $f(x)$ is **continuous** at $x = a$ if:

1. $\lim_{x \to a} f(x)$ exists.
2. $f(a)$ is defined.

3. $\lim_{x \to a} f(x) = f(a)$.

Essentially, this means that you can take a pen, approach $(a, f(a))$ from either the positive or negative side, and go through it without picking up the pen.

Many functions are continuous at all points in their domain, such as polynomials, exponential functions, or sine and cosine functions. Some notable exceptions include functions such as trigonometric functions $\tan, \sec, \csc, \cot$; and rational functions such as $\frac{1}{x}$. Many of the courses we will be dealing with in this class will be continuous so make sure to be able to differentiate a continuous function from one that is not.

0.2 Derivatives

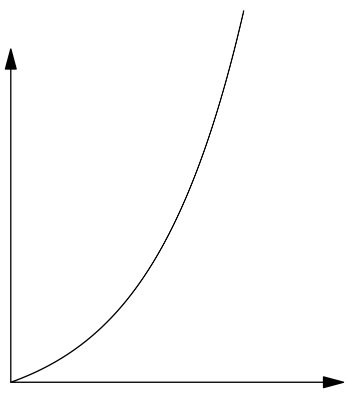
0.2.1 Definition

For a linear function $y = mx + b$, we know that the slope, m, gives the rate of change of the function. But how can we generalize this notion of rate of change to x^2, $\sin(x)$ or other functions, where the rate of change is not constant?

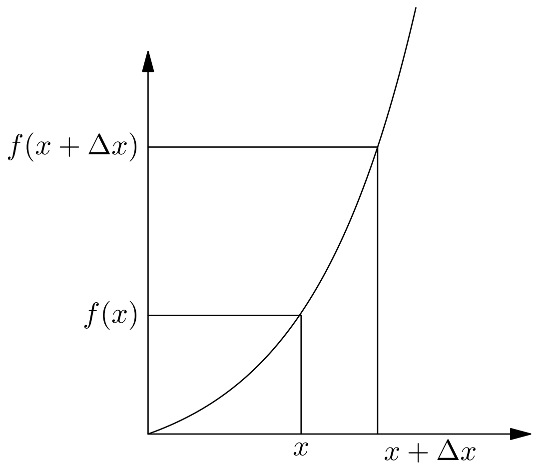
To find the instantaneous rate of change at a point $x = a$ is equivalent to finding the slope of the tangent line to the graph at $x = a$. Why does this work? The tangent line intersects the graph locally at one point only, but it can be thought of as the line through two infinitesimally close points on f. This means the tangent line's slope matches the slope through these two infinitesimally close points, which is really just the slope locally at that one point.

Since the value of this slope might be constantly changing, the rate of change of $f(x)$ would be a function of x as well. This is what we call the derivative of $f(x)$ with respect to x.

Let's say we are given a function $f(x)$. Mathematically, how should we define the derivative of $f(x)$? Consider the graph of an arbitrary function:



Let's consider how we calculate the average slope over an interval:



The slope of any line is given as $\frac{\text{rise}}{\text{run}}$, so thus the average slope over the this interval $(x, x+\Delta x)$ is given by $\frac{f(x)-f(x+\Delta x)}{\Delta x}$. However, this is an *average* and not very accurate. To find the slope at a specific instant, we take the limit as the interval approaches zero. In other words, we take the limit as $\Delta x \to 0$. Over this "infinitesimally" small interval, the function acts like it is linear. This is what motivates the following definition of a derivative.

Definition 0.2. For a function $f(x)$, the **first derivative** of $f(x)$ with respect to x is given by

$$\frac{\mathrm{d}}{\mathrm{d}x} f(x) = \lim_{\Delta x \to 0} \left(\frac{f(x+\Delta x) - f(x)}{\Delta x} \right)$$

There are many ways of denoting the derivative of a function. The derivative of a function $f(x)$ can also be denoted by any of the following: $f'(x), f', \dot{f}$, or $\frac{df}{dx}$.

Concept. The key takeaway here is that the derivative of a function $f(x)$ gives the rate of change of $f(x)$ as x changes.

0.2.2 Differentiation Rules

We will now give basic rules to evaluate the derivative of various functions. We will not give proofs, since they are rather computational and come from evaluating the limit definition of a derivative, but they are easily accessible online. For the following identities, let f and g be functions of x. Then:

1. Differentiation is linear. This means that for a constant c,
$$(c \cdot f)' = c \cdot f',$$
and
$$(f+g)' = f' + g'.$$
2. The Power Rule:
$$(f^n)' = n \cdot f^{n-1} \cdot f'.$$
When $f(x) = x$, we have the simplified form
$$(x^n)' = nx^{n-1}.$$
3. The Product Rule:
$$(fg)' = fg' + f'g.$$

4. The Chain Rule:
$$(f(g(x)))' = f'(g(x))g'(x).$$

The chain rule also has an alternate form,
$$\frac{df}{dx} = \frac{df}{dg} \cdot \frac{dg}{dx}.$$

This has an important application, as we will now see.

Concept (Implicit Differentiation). So far, we have only taken the derivative of a function $f(x)$. However, sometimes it is burdensome to express a function in the form $f(x) = \cdots$. In this case, we can simply take the derivative with respect to x using the chain rule.

To illustrate this, consider the following example.

Example 0.3. For the graph of
$$x^3 + y^2 = 64,$$
give $\frac{dy}{dx}$ as a function of x and y.

Solution. In this case, y is not a function of x. However, we can still find the slope through differentiation. Taking the derivative of both sides gives
$$\frac{d}{dx}\left(x^3 + y^2\right) = \frac{d}{dx}(64),$$
and applying the chain rule gives
$$3x^2 + 2y\frac{dy}{dx} = 0.$$
Now rearranging gives that
$$\frac{dy}{dx} = -\frac{3x^2}{2y}.$$

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The derivatives of sine and cosine will also show up in physics, and as such are rather important. The differentiation rules are as follows. For a constant a,
$$\frac{d}{dx}\sin(ax) = a\cos(ax)$$
$$\frac{d}{dx}\cos(ax) = -a\sin(ax).$$

The differentiation rules for logarithms and exponentials don't appear particularly often in physics, but we will list them here as well.
$$\frac{d}{dx}\left(\log_a(f(x))\right) = \frac{f'(x)}{f(x)\ln a}$$
$$\frac{d}{dx}\left(a^{f(x)}\right) = f'(x)a^{f(x)}\ln a.$$

This concludes our summary of basic differentiation rules. In a typical calculus class, a lot of time is spent memorizing and applying these rules. For our purposes, however, this is generally unnecessary. The most important differentiation rules to know are the power rule and the rules for sine and cosine. The chain rule is also helpful, though mainly for implicit differentiation.

0.3 Optimization

As you might have noticed, in our definition of derivative we used the term *first derivative*. It is typically implied that the term "derivative" refers to the first derivative. However, higher-order derivatives do exist and are important as well.

Definition 0.4. For a function $f(x)$, define the ***k*-th derivative** of $f(x)$ with respect to x (denoted by $f^{(k)}(x)$) as the first derivative of the $(k-1)$th derivative of $f(x)$:

$$f^{(k)}(x) = \frac{d}{dx}\left(f^{(k-1)}(x)\right)$$

For example, the second derivative of a function $f(x)$ is the derivative of $f'(x)$.

Second-order derivatives show up frequently in physics, but derivatives of even higher orders are rarely useful. The second derivative of $f(x)$ is often denoted by $f''(x)$, f'', $\ddot{f}$, or $\frac{d^2f}{dx^2}$.

Recall that $f'(x)$ is the rate of change of $f(x)$, and that $f''(x)$ is the rate of change of $f'(x)$. This will intuitively give us the following results.

Concept.

1. A function $f(x)$ is increasing when $f'(x) > 0$, decreasing when $f'(x) < 0$, and constant when $f'(x) = 0$. *For x in the domain of f, points $(x, f(x))$ where $f'(x) = 0$ or where $f'(x)$ is undefined are called* ***critical points***.
2. A function $f(x)$ is convex when $f''(x) > 0$, concave when $f''(x) < 0$, and linear when $f''(x) = 0$. *Points where $f''(k) = 0$ are called* ***inflection points***.

For those unfamiliar with convex and concave functions, here is a simplified explanation. The term convex (also called concave up) refers to a function whose graph is "bowl-shaped", while the term concave (also called concave down) refers to a function whose graph is "dome-shaped."

The reason critical points are so interesting is that all local minimums or local maximums of a function must occur at a critical point (though not all critical points must be local extrema). Note that if $f'(x)$ changes from positive to negative, then $f(x)$ has changed from increasing to decreasing. There must be a peak at some point; this is in fact the point at which $f'(x) = 0$. Hence, we have the following result:

Concept.

1. A function $f(x)$ has a local minimum at $x = c$ if $f'(x)$ changes from negative to positive at $x = c$.
2. A function $f(x)$ has a local maximum at $x = c$ if $f'(x)$ changes from positive to negative at $x = c$.

We'll do a quick example to see how this helps with optimization problems.

Example 0.5. Olenna has 1000 feet of fence with which she wants to make a rectangular garden. One side of the fence will lean against a castle wall and will not need to be fenced. Find the dimensions of the garden that will maximize its area.

Solution. Let the dimensions of the garden by $w \times \ell$, where (without loss of generality), we let ℓ be the length of the unfenced side. Then we have the following equation:

$$2w + \ell = 1000$$
$$\Rightarrow \ell = 1000 - 2w.$$

Hence, the area of the garden can be expressed as a function of w:

$$A = w\ell = w(1000 - 2w).$$

Note that when $A'(w) = 0$, this is a critical point, which will give us the value of w for our desired relative maximum. Setting A' equal to zero gives

$$A' = 1000 - 4w = 0$$
$$w = 250, \ell = 500.$$

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Similarly, any relative maxima or minima of $f'(x)$ must occur at an inflection point of $f(x)$. However, inflection points are also important because any change in the concavity of $f(x)$ must occur at an inflection point.

Maximizing and minimizing quantities are crucial in many physics problems, so make sure you know how. This sums up the basics of using calculus for optimization problems. However, differentiation has another important application, as we will see in the following section.

0.4 Related Rates

In problems where there are multiple variables which are functions of a single variable, typically time(t), the relationship involving their rates of change can be obtained by differentiating with respect to that common variable. We'll now do some examples to show what we mean.

> **Example 0.6.** Loras is baking a spherical tart, the volume of which is expanding at a rate of 24 cubic centimeters per minute. At what rate is the radius r changing when $r = 4$ ft?

Solution. Note that the problem tells us $\frac{dV}{dt} = 24$, and wants us to evaluate $\frac{dr}{dt}$ when $r = 4$. The volume of the tart as a function of radius is given by

$$V = \frac{4}{3}\pi r^3,$$

as with any sphere. Taking the derivative with respect to time, we have that

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}.$$

Hence, substituting gives us that

$$\frac{dr}{dt} = \frac{3}{8\pi} \text{ ft/min}.$$

■

The key here is to differentiate with respect to t, and substitute the rate we are already given.

We'll now do another example that might seem initially like a physics problem, but is again solved easily using calculus.

Example 0.7. A 13 foot ladder is leaning against a castle wall, and the top of the ladder is sliding downward at a rate of 0.5 m/s. How quickly is the base of the ladder sliding when the base is 12 m away from the castle wall?

Solution. Let x denote the horizontal distance from the wall to the base of the ladder, and let y denote the vertical distance from the floor to the top of the ladder. The length of the ladder is 13 feet, so by the Pythagorean Theorem, we have that

$$x^2 + y^2 = 169.$$

Taking the derivative with respect to t, we have that

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0.$$

Now recall the conditions give $x = 12$ and $\frac{dy}{dt} = -.5$. By the Pythagorean Theorem again, we see that at this instant $y = 5$. Hence, substituting gives that

$$\frac{dx}{dt} = \frac{5}{24} \text{ m/s}.$$

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0.5 Integrals

As we have seen, we can obtain the derivative of a function through differentiation. However, given the derivative of a function, can we determine the original function?

The answer is not quite. Remember that the derivative is just the slope of a graph. If we shift the graph vertically, the derivative will not change (the derivative of a constant is zero). However, given a specific point on the original graph, we can determine the original function. This process is called integration.

Definition 0.8. For a function $f : [a, b] \to \mathbb{R}$, we call a differentiable function $F : [a, b] \to \mathbb{R}$ an anti-derivative of f if $F'(x) = f(x)$ for all $x \in (a, b)$.

Definition 0.9. Define the **indefinite integral**

$$\int f(x)\, dx$$

as the set of all anti-derivatives of f. Since all anti-derivatives of f differ only by a constant, this is generally notated as

$$\int f(x)\, dx = F(x) + c,$$

where F is an anti-derivative of f, and c represents an arbitrary constant. You may be asking, why do we have c there? The main thing is, that when you differentiate $F(x) + c$ back, you'll notice that

$$\frac{d}{dx}F(x) + c = \frac{d}{dx}\int f(x)dx = f(x)$$

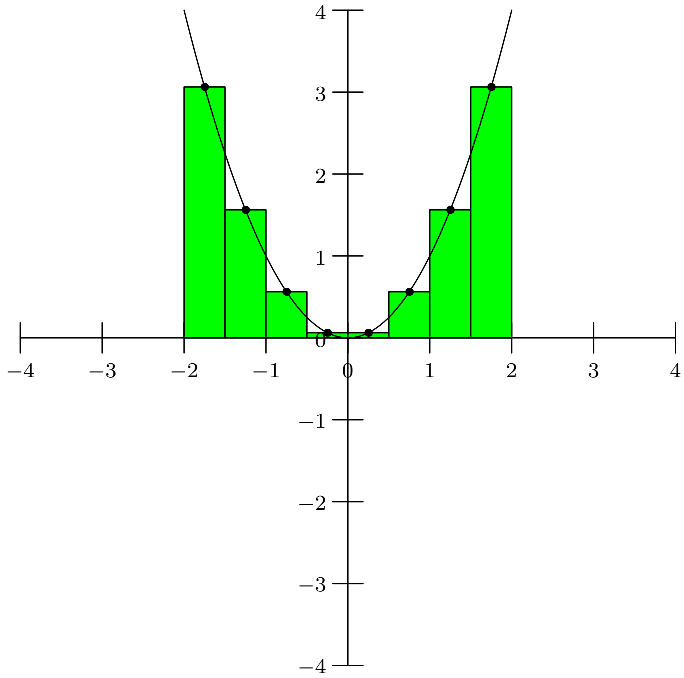
since the derivative of a constant c is simply the slope of the line $y = c$ which is just zero. If we have initial values, we can find the value of c which can help us in real life models.

For example, we have that the indefinite integral of x is

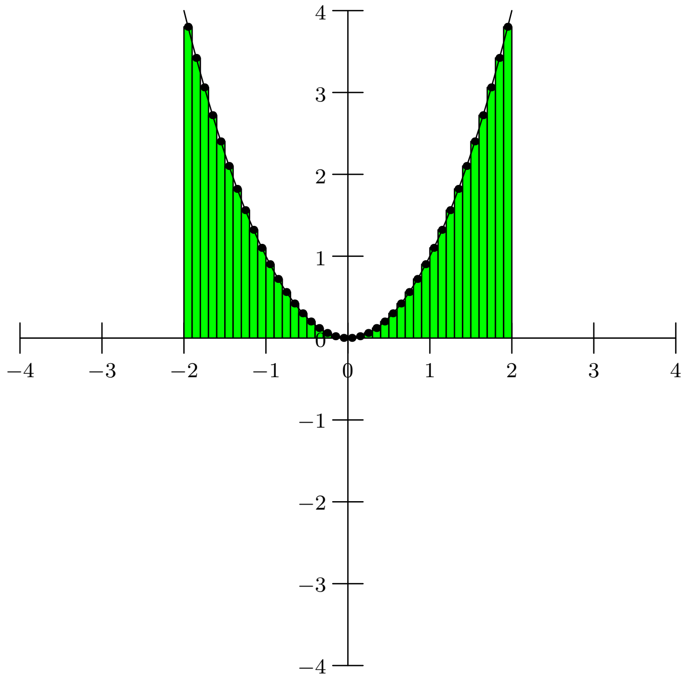
$$\int x \, dx = \frac{1}{2}x^2 + C.$$

To determine a function from its derivative, we can evaluate the indefinite integral, and determine c by substituting an arbitrary point satisfied by the function.

Another useful form of integration is called the **definite integral**. The motivation for the definite integral is to take the signed area under the curve. One way of approximating the area is using rectangles:



This is not very accurate, but by shrinking the intervals of each rectangle, we can improve the approximation. In fact, by shrinking the time intervals to be infinitesmally small, we can obtain the exact area.



Definition 0.10. Given a function $f(x)$ continuous on the interval $[a, b]$, divide $[a, b]$ into n equal subintervals of width Δx and from each interval choose a point x_i. Then we define the definite integral of $f(x)$ from a to b as

$$\int_a^b f(x)\, dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x.$$

Concept. The key takeaway is that the definite integral of $f(x)$ evaluates the signed area under the curve of the graph of $f(x)$. Areas under the x-axis are evaluated with a negative sign.

Let us look at a table of important integral identities in calculus now:

$\int du = u + c$	$\int \frac{du}{u} = \ln \lvert u\rvert + c$
$\int a^u du = \left(\frac{1}{\ln a} a^u\right) + c$	$\int u^n du = \frac{u^{n+1}}{n+1} + c, \qquad n \neq -1$
$\int \cos u \quad du = \sin u + c$	$\int \sin u \quad du = -\cos u + c$
$\int \tan u \quad du = -\ln \lvert\cos u\rvert + c$	$\int \sec u \quad du = \ln \lvert\sec u + \tan u\rvert + c$
$\int e^u du = e^u + c$	$\int \sec^2 u \quad du = \tan u + c$
$\int \csc^2 u \quad du = -\cot u + c$	$\int \csc u \cot u \quad du = -\cot u + c$

Let us look at a quick example of where integrating will be beneficial.

Example 0.11 (2019 Excellence in Mathematics Individual Test)**.** Suppose $\frac{dy}{dt} = 20t - 0.1t^2$, where y represents the quantity of bolts produced, in thousands, after t hours of manufacturing. If $y(0) = 100$, determine the total number of bolts, in thousands, produced over the interval $2 \leq t \leq 5$. Round your answer to the nearest hundredth.

Solution. We simply take the definite integral

$$dy = \int_2^5 (20t - 0.1t^2) dt \implies y = 206.1$$

to see 206.1 bolts were produced on the time interval. ■

We now state without proof the following important theorem which allows us to compute definite integrals.

Theorem 0.12 (Fundamental Theorem of Calculus)**.** *This theorem consists of two parts:*

(a) *Let f be a continuous real-valued function on $[a, b]$, and let F be a function on $[a, b]$ defined by*

$$F(x) = \int_a^x f(t)\, dt.$$

Then it holds that F is an anti-derivative of f on $[a, b]$.

(b) *Let f be a real-valued function on $[a, b]$, and F an antiderivative of f on $[a, b]$. Then* *it holds that*

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

Hence, just as we can calculate indefinite integrals with anti-derivatives, we can also calculate definite integrals using anti-derivatives.

0.6 Differential Substitutions

In physics we sometimes will be given integrals that are difficult to integrate, so it is important to know which substitutions we can make. For instance, if we have

$$A = \int y\,dx,$$

we can write this as

$$dA = y\,dx.$$

We will now give a list of substitutions for differentials that are often useful in physics:

1. $$dA = x\,dy.$$

   This substitution comes as a symmetry of $dA = y\,dx$ by switching x and y.
2. For a circular shape, it might help that we use the substitution

   $$dA = 2\pi r\,dr,$$

   where r is distance from the origin. This substitution comes from considering an infinitesimal change in area change as adding a circular ring of radius r and infinitesimal thickness dr (which will hence have an area of $2\pi r \cdot dr$).
3. In general, we have the volume differential is equal to

   $$dV = A\,dz,$$

   where A is the cross-sectional area and dz is an infinitesmal thickness.
4. For a cylindrical shape of radius R length ℓ, we also have the volume differential

   $$dA = 2\pi r\ell\,dr = \pi R^2 dz.$$

   The first substitution is a generalization of the circular volume differential (integrating from r), whereas the second substitution comes from $dV = A\,dz$ (integrating along the long axis of the cylinder).
5. Similarly, for spherical shapes have that the volume differential is equal to

   $$dV = 4\pi r^2\,dr.$$

   This comes from considering the change in volume as adding a spherical shell of radius r and infinitesimal thickness dr.

We will now do an example.

Example 0.13. A circular disk of radius R has its center located at the origin $(0, 0)$. The the circular disk has density given by the function $x^2 + y^2$. What is the mass of the disk?

Solution. We wish to compute

$$M = \int dm.$$

Note that in this case, density is dm/dA. Hence, we have

$$\frac{dm}{dA} = x^2 + y^2 \implies dm = (x^2 + y^2)\, dA.$$

It is easier for us to now convert to polar coordinates. We have

$$\int (x^2 + y^2)\, dA = \int r^2 dA = \int_0^R r^2 \cdot 2\pi r\, dr = \frac{\pi}{2} R^4.$$

■

0.7 Differential Equations

Throughout the course, we will have to solve a number of **differential equations** to find the time-dependence of physical quantities. A differential equation is one that involves derivatives (which in our case, will mostly be with respect to time) of the variable we're solving for. Here are a few examples of differential equations:

1. $\frac{dy}{dx} = 3$
2. $\frac{dy}{dx} = x \sin x$
3. $x\frac{dy^2}{dx^2} = \frac{1}{\ln y}$

There are many types of differential equations, with some more complicated than others. *Solving* a differential equation usually entails finding the dependence of one of the variables on another. This is more easily said than done however, as often times solving a differential equation can become *extremely hard* and at times even impossible. In this course however, we'll only be dealing with a very small subset of differential equations. Namely, we'll only be dealing with **ordinary differential equations** (ODE), a differential equation with one or more functions of a *single* variable, with all derivatives relative to that same variable. All the differential equations in the list above are ordinary differential equations (Note that we do not have any partial derivatives and all derivatives are with respect to x).

Now, in a typical differential equations course, you'll general learn a number of tips and tricks to solving various types of ODEs. In this course however, you'll only really have to be able to solve differential equations that are *separable*. The solutions to all other types of differential equations will be given.

Sometimes the differentials in the differential equations can be *separated* and then integrated which results in a solution: for example, consider the differential equation

$$\frac{dy}{dx} = -\frac{x}{ye^{x^2}}$$

Note that we can separate the xs and ys to their own side of the equation so that we get

$$ydy = \frac{x}{e^{x^2}}dx$$

We now note that we can integrate both sides which gives

$$\int y dy + C_1 = \int \frac{x}{e^{x^2}} dx$$

$$\frac{1}{2}y^2 = \frac{1}{2}e^{-x^2} + C_2$$

for some constants C_1, C_2. Note however, that this gives us a solution to the differential equation: namely by multiplying both sides by 2 and then squaring, we get that solutions of the form

$$y = \pm\sqrt{e^{-x^2} + C}$$

where C is some constant forms the solution set to the differential equation.

For more information on separable differential equations see An Introduction to Separable Differential Equations.

We'll encounter other separable equations often in the future so make sure you know how to solve them when presented with one.

0.8 Taylor Approximations and Series

The final application of calculus we present is in approximations. Consider the following theorem, which we state without proof.

> **Theorem 0.14.** *An infinitely differentiable function $f(x)$ can be represented by the sum*
>
> $$f(x) = \frac{f(a)}{0!}(x-a)^0 + \frac{f'(a)}{1!}(x-a)^1 + \frac{f''(a)}{2!}(x-a)^2 + \cdots.$$

The motivation behind this theorem is that substituting the nth derivative of $f(x)$ (denoted $f^{(n)}(x)$) for $f(x)$ is equivalent to differentiating the right-hand side n times. Since this must hold for any $n \geq 1$, we obtain this series as the sum. This sum is called the **Taylor series** centered at $x = a$, and when centered at $x = 0$, it is also called the **Maclaurin series**.

The Maclaurin series of $(1-x)^{-1}$, for example, is given by

$$(1-x)^{-1} = 1 + x + x^2 + x^3 + \cdots$$

The reason these series expansions are so useful is that under certain constraints, they allow us to approximate more complicated functions. For instance, if $x \ll 1$, then we can approximate that $1/(1-x) \approx 1+x$, since the other terms are all exponentially smaller in magnitude. Similarly, we have the binomial expansion:

$$(1+x)^n = 1 + nx + \binom{n}{2}x^2 + \cdots + nx^{n-1} + x^n.$$

As with the previous Maclaurin series, for small values of $x \ll 1$, we can approximate $(1+x)^n \approx 1 + nx$.

Two other Maclaurin series that give useful approximations in physics are

$$\sin(x) = \frac{1}{1!}x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \cdots$$

$$\cos(x) = \frac{1}{0!} - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 + \cdots + (-1)^n \frac{x^{2n}}{(2n)!} + \cdots$$

Hence for small values of $x \ll 1$, we can approximate that $\sin(x) \approx x - \frac{1}{6}x^3$ and $\cos(x) = 1 - \frac{1}{2}x^2$. Sometimes, it might even be necessary to approximate $\sin(x) \approx x$ and $\cos(x) \approx 1$. In this case, tangent can also be approximated by $\tan(x) \approx x$. This specific approximation is referred to as the *small-angle approximation* which roughly holds when $x < 10^{\text{deg}}$.

In summary, for small values of $x \ll 1$, we have the following:

$$\begin{aligned} \frac{1}{1-x} &\approx 1 + x \\ (1+x)^n &\approx 1 + nx \\ \sin x &\approx x \\ \cos x &\approx 1 - \frac{x^2}{2}. \end{aligned}$$

All four of these approximations will be useful in physics problems and allow us to approximate otherwise chaotic calculations, particularly in the presence of nasty integrals that would be otherwise be difficult (or at times impossible) to solve. They will also make a number of our formulas "nicer" although we still need to make sure we uphold the conditions for which these approximations hold.